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New mathematical tools for the Lagrangian description of atmospheric dynamics

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Abstract

Applications of the mathematical theory of rearrangements to atmosphere and ocean dynamics are discussed. Results in the existing literature, mainly for two-dimensional incompressible flow, are reviewed. We prove that a state of geostrophic and hydrostatic balance minimises the energy subject to conservation of absolute momentum and potential temperature, and show that a steady state of a semigeostrophic flow can be characterised as a stationary point of the energy subject to rearrangements of the potential vorticity. By restricting the class of rearrangements to those consistent with the evolution equations, it is suggested that the enstrophy cascade may be prevented in semigeostrophic flow.

1 Introduction

The description of weather systems in the atmosphere has always drawn a great deal on a Lagrangian description of the flow. Obvious examples are the description of synoptic developments in terms of air masses, the parcel theory of convection, and the description of the dynamics of precipitation systems in terms of conveyor belts, (e.g. Browning and Roberts(1994)). The importance of Lagrangian concepts in dynamical understanding has been highlighted in recent years by the extensive use of potential vorticity diagnostics, (Hoskins et al. (1985)). The power of Lagrangian methods in numerical simulation has been demonstrated by contour dynamics algorithms (Dritschel (1988ab)). Unfortunately, however, much theory still uses a Eulerian description of the flow because more mathematical tools are available in that case. For instance, the analysis of numerical algorithms for advection is very incomplete because, in a Eulerian sense, the problem is nonlinear unless the advecting velocity is uniform. In general, little can be said about the solution of nonlinear problems, apart from verifying global conservation properties, so that the analysis stops with the linear case. However, the process being described is quite simple, the advected quantity is transported along trajectories with no change in value. This is exploited by Lagrangian numerical methods, and should be susceptible to analysis.

A concept that arises naturally when considering the Lagrangian description of incompressible fluid motion is that of a rearrangement. Let σ represent a quantity, associated with a fluid, which is conserved following the motion of a fluid particle, such as potential vorticity in adiabatic flow. Assume we are working in a bounded region. As time advances, the fluid particles will, in general, exchange their spatial positions, but each

particle retains its original value of σ. Then we can say that, for any given time t, σ considered as a function $\sigma(\mathbf{x}, t)$ is a *rearrangement* of the initial function $\sigma(\mathbf{x}, 0)$, where $\mathbf{x}$ is the typical Eulerian position vector. Alternatively we say that σ is *rearrangement preserved* by the flow. In this paper, we define this concept precisely and then show how some tools available from rearrangement theory can be used in analysis of atmospheric (or oceanic) flows. Further mathematical details and proofs are given by Douglas (1996).

Three practical results are illustrated. The first is the characterisation of minimum energy states subject to conservation of properties following particles. It is shown that it is necessary to consider whether the extreme state can be reached by mixing the fluid, in which case the use of conservation following particles may not be appropriate. The second is the behaviour of the time evolution of a flow subject to (potential) vorticity conservation. Again it is crucial whether the long time evolution approximates mixing, equivalent to the enstrophy cascade of two-dimensional turbulence, or whether this is prohibited. The third is the study of the stability of steady states, following the work of Nycander (1995). If we work in a class of perturbations which are rearrangements of the potential vorticity, we can show that steady states can be characterised as stationary points of the energy. This class of perturbations is a subclass of all possible perturbations and is chosen to be consistent with the time dependent dynamics, which conserve potential vorticity. The result is not true if all perturbations are allowed.

2 Atmospheric models and rearrangements

As shown by Hoskins *et al.* (1985), many balanced models of the atmosphere can be described in terms of the Lagrangian conservation of a potential vorticity q by an equation

$$\frac{Dq}{Dt} = 0 \tag{1}$$

and an 'invertibility principle'

$$H(p, \mathbf{u}, \theta) = q \tag{2}$$

allowing pressure, velocity, and potential temperature to be derived. H is typically an elliptic operator which may be nonlinear. Boundary conditions are also required. Conditions, e.g. non- negativity, may have to be imposed on q to allow (2) to be solved uniquely. Specific examples are

i) Barotropic vorticity equation

Assume this is to be solved in a region Ω with boundary $\partial\Omega$. (1) takes the form

$$\frac{D}{Dt}(\zeta + f) = 0 \tag{3}$$

where ζ is the relative vorticity and f the Coriolis parameter. (2) takes the form

$$\begin{aligned} \nabla^2\psi &= \zeta \text{ on } \Omega \\ \psi &= 0 \text{ on } \partial\Omega \\ (v, -u) &= \nabla\psi \end{aligned} \tag{4}$$

(4) can be solved for any values of ζ. There is no solvability condition.

ii) Quasi-geostrophic theory

Following Hoskins et al (1985), (2) takes the form

$$\nabla_h^2\psi + f_0^2 \frac{\partial}{\partial p}\left(N^{-2}\frac{\partial\psi}{\partial p}\right) = q - f \tag{5}$$

with boundary condition

$$f_0 \frac{\partial \psi}{\partial p} = -R(\theta - \theta_{\text{ref}}) \tag{6}$$

at $p = 0, p_0$. Here, subscript h represents a derivative at constant p, N is the Brunt-Vaisala frequency, and ψ the geostrophic streamfunction. θ_{ref} is a reference potential temperature.

$$q = f + \mathbf{k}.\nabla_p \times \mathbf{v} + f_0 \frac{\partial}{\partial p}\left(\frac{\theta - \theta_{\text{ref}}}{d\theta_{\text{ref}}/dp}\right) \tag{7}$$

q is conserved following the horizontal geostrophic motion. (5) can be solved for ψ on a region $\Omega \times [0, p_0]$ with boundary $\partial\Omega$ given the additional boundary condition $\psi = 0$ on $\partial\Omega$. The remaining fields can then be calculated from the geostrophic relation $\mathbf{v} = \mathbf{k} \times \nabla\psi$ and the linearised hydrostatic relation $f_0 \partial\psi/\partial p = -R(\theta - \theta_{\text{ref}})$. Since (5) is linear, there is no solvability condition. In particular, there is no distinction between situations where $(q - f)$ is positive or negative.

iii) Semigeostrophic theory on an f plane

This is most simply expressed in the pseudo-height coordinate of Hoskins (1975). Then (2) takes the form that the Hessian of P

$$\det \begin{pmatrix} \frac{\partial^2 P}{\partial x^2} & \frac{\partial^2 P}{\partial x \partial y} & \frac{\partial^2 P}{\partial x \partial z} \\ \frac{\partial^2 P}{\partial y \partial x} & \frac{\partial^2 P}{\partial y^2} & \frac{\partial^2 P}{\partial y \partial z} \\ \frac{\partial^2 P}{\partial z \partial x} & \frac{\partial^2 P}{\partial z \partial y} & \frac{\partial^2 P}{\partial z^2} \end{pmatrix} = q \tag{8}$$

where $P = \phi + \frac{1}{2}(x^2 + y^2)$, ϕ is the geopotential, and the geostrophic wind and potential temperature are given by $\nabla\phi = (fv_g, -fu_g, g\theta/\theta_0)$. In order to solve (8), define $\rho = \frac{1}{q}$ and $R = \mathbf{x}.\mathbf{X} - P$ where the geostrophic and isentropic coordinates $\mathbf{X}$ are given by

$$\mathbf{X} = (x + v_g/f, y - u_g/f, g\theta/(f^2\theta_0)) \tag{9}$$

Then, following Cullen and Purser(1989),

$$\det \begin{pmatrix} \frac{\partial^2 R}{\partial X^2} & \frac{\partial^2 R}{\partial X \partial Y} & \frac{\partial^2 R}{\partial X \partial Z} \\ \frac{\partial^2 R}{\partial Y \partial X} & \frac{\partial^2 R}{\partial Y^2} & \frac{\partial^2 R}{\partial Y \partial Z} \\ \frac{\partial^2 R}{\partial Z \partial X} & \frac{\partial^2 R}{\partial Z \partial Y} & \frac{\partial^2 R}{\partial Z^2} \end{pmatrix} = \rho \tag{10}$$

can be solved in a convex region Ω in (x, y, z) if the right hand side is non-negative subject to the boundary condition

$$\nabla_x R \in \Omega \tag{11}$$

The form of (10) guarantees that R is a convex function. The situation is now different from quasi-geostrophic theory in that the equations cannot be solved for negative potential vorticity.

The solution of all these models can be described as rearrangements of the initial potential vorticity distribution. Since no new values can be created, or existing values destroyed, there is a significant constraint on the resulting flows. The rearrangement is determined by calculating an advecting velocity field using the invertibility principle. This procedure constrains the velocity field and thus gives information about the type of rearrangement that can be reached from given initial data. These ideas have been exploited mathematically for case (i) by Kato (1967) and case (ii) by Bennett and Kloeden (1981), amongst others. These results are summarised in section 6.

Another type of application is the idea of minimising a function such as the energy with respect to rearrangements. Shutts *et al* (1988) illustrated that insight could be obtained into many processes in the atmosphere by using the ideas of minimising the energy of a fluid, subject to certain conservation laws. The simplest example is the rearrangement of a vertical column of fluid where the buoyancy, B, of each fluid particle is fixed. Thus we

specify the initial buoyancy as a function $B_0(z)$, and seek that function B(z) which is a rearrangement of B_0 and minimises the energy $\int_0^h -Bzdz$. The minimum energy state is that with the heaviest fluid at the bottom and a monotonic increase of buoyancy with height. Other examples treated by Shutts *et al* were in two-dimensional cross-sections (x, z), where the potential temperature and the y component of absolute momentum $(v + fx)$ are simultaneously conserved, since there are no pressure gradients in the y direction in such problems. It is shown that two-dimensional semigeostrophic solutions correspond to a sequence of such minimum energy states, where the time evolution of potential temperature and absolute momentum is prescribed.

Cullen *et al* (1987) showed that solutions of the three dimensional semi-geostrophic equations could be interpreted this way, even in spherical geometry. If (X, Y, Z) are the coordinates defined in (9), the solution is found by predicting (X, Y, Z) on fluid particles using the equations

$$\begin{aligned} \dot{X} &= f(y - Y) \\ \dot{Y} &= f(X - x) \\ \dot{Z} &= 0 \end{aligned} \tag{12}$$

and then finding the location of the particles as the minimum energy rearrangement $\mathbf{X}_0 = \mathbf{X}_0(\mathbf{x})$, where the geostrophic energy is that which is minimised. Thus the solution can be characterised as a sequence of minimum energy states.

A different use of the idea of energy minimisation is contained in Vallis (1992). He shows that geostrophic and hydrostatic balance can be characterised as the minimum energy state of the atmosphere with a given potential vorticity distribution. Roulstone and Sewell (private communication) show that if the extremisation is carried out with respect

to a given distribution of potential vorticity on fluid particles, then the resulting state is a steady solution of the primitive equations. Similar ideas were used by Nycander (1995) in a study of the stability of steady states of two-dimensional incompressible flow. He was able to characterise states where the energy perturbation is positive or negative definite, and thus establish stability.

3 Basic definitions and results from rearrangement theory

3.1 Rearrangements of scalar valued functions

We begin by giving an intuitive idea of when a function is a rearrangement of another function. Let Ω be a bounded region in $\mathbb{R}^n$, and let $f : \Omega \to \mathbb{R}^d$ be a function. Imagine that Ω is a continuum of infinitesimal particles, so that $n = 2$ or 3 in general. Suppose we exchange the particle positions, with each particle retaining its value of f. This yields a new function $g = g(\mathbf{x})$, say, which we call a *rearrangement* of $f = f(\mathbf{x})$. The idea of rearranging a function can be applied to both scalar and vector valued functions. The latter case is important if there is more than one quantity which is conserved by the flow.

We give a simple example of two scalar functions which are rearrangements of each other. Let $f(x) = x$ for each $x \in [0, 1]$, and let g be defined by

$$g(x) = \begin{cases} 1 - 2x & \text{if } x \in [0, 1/2], \\ 2x - 1 & \text{if } x \in [1/2, 1]. \end{cases} \tag{13}$$

For each $0 < \alpha < 1$,

$$\int_0^1 (f(x) - \alpha)_+ dx = \frac{1}{2}(1 - \alpha)^2 = \int_0^1 (g(x) - \alpha)_+ dx \tag{14}$$

where the subscript $+$ denotes the positive part of the function. (14) says that the area of the region which is bounded above by the graph of f and bounded below by the line $y = \alpha$ is the same as the area of the corresponding regions of g. This is illustrated in Figure 1: the total shaded areas are the same. If $\alpha \geq 1$, then $\int_0^1 (f(x) - \alpha)_+ dx = 0 = \int_0^1 (g(x) - \alpha)_+ dx$. Combining the above results, $\int_0^1 (f(x) - \alpha)_+ dx = \int_0^1 (g(x) - \alpha)_+ dx$ for each positive α.

We will use this property to make a precise definition of when f and g are *rearrangements* of each other.

Following Eydeland, Spruck and Turkington (1990), let $f, g : \Omega \to \mathbb{R}$ be two non–negative integrable functions i.e $\int_\Omega f(\mathbf{x})d\mathbf{x} < \infty$, $\int_\Omega g(\mathbf{x})d\mathbf{x} < \infty$. f is a *rearrangement* of

g if

$$\int_\Omega (f(\mathbf{x}) - \alpha)_+ d\mathbf{x} = \int_\Omega (g(\mathbf{x}) - \alpha)_+ d\mathbf{x} \tag{15}$$

for each $\alpha > 0$.

In order to make it easier to extend the concept to vector-valued functions, it is more convenient to work with an alternative definition. This is expressed in terms of the measures of sets on which f and g take specified values. The measure of a set indicates its size. In $\mathbb{R}$ the measure could be length, in $\mathbb{R}^2$, area, in $\mathbb{R}^3$, volume. Returning to the example of Fig.1, consider the sets $\{x \in [0,1] : f(x) \in [1/2,1]\}$ and $\{x \in [0,1] : g(x) \in [1/2,1]\}$. The former set is $[1/2,1]$ and the latter $[0,1/4] \bigcup [3/4,1]$. These two sets have the same Lebesgue measure (or length). This is illustrated in Figure 2. A general statement of this property is contained in the following definition.

Definition Let non–negative $f, g : \Omega \to \mathbb{R}$ be integrable functions, where $\Omega \subset \mathbb{R}^n$ is a bounded region. f and g are rearrangements if

$$\mu(\{\mathbf{x} : f(\mathbf{x}) \in B\}) = \mu(\{\mathbf{x} : g(\mathbf{x}) \in B\}) \tag{16}$$

for each Borel set $B \subset \mathbb{R}$, where μ denotes n–dimensional Lebesgue measure. By restricting our definition to Borel sets of $\mathbb{R}$ (which excludes some pathological subsets of $\mathbb{R}$), we exclude sets whose size cannot be calculated, so that (16) is always well defined.

This definition can be shown to be equivalent to the one given earlier. Another equivalent formulation which appears in the literature (see Burton (1987) for example) states that for f and g as in the above definition, f is a rearrangement of g if

$$\mu(\{\mathbf{x} : f(\mathbf{x}) \geq \alpha\}) = \mu(\{\mathbf{x} : g(\mathbf{x}) \geq \alpha\}) \tag{17}$$

for each positive α.

We now introduce the concept of a *set* of rearrangements of a given function. A function $f = f(x)$ has more than one rearrangement (unless it is constant). Two different

rearrangements of the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = x$ are illustrated in Figure 3. We write $R(f_0)$ to denote the set of rearrangements of a prescribed function f_0.

3.2 Rearrangements of vector valued functions

We introduce the concept of rearrangement of vector valued functions by an example. Define a function f on the unit square $[0,1] \times [0,1]$ by

$$f(\mathbf{x} = (x,y)) = \begin{cases} (1,1) & \text{if } 1/2 \le x \le 1, 1/2 \le y \le 1 \\ (0,0) & \text{otherwise,} \end{cases} \tag{18}$$

and another function g by

$$g(\mathbf{x} = (x,y)) = \begin{cases} (0,0) & \text{if } 1/8 < y < 7/8, \\ (1,1) & \text{if } y \le 1/8 \text{ or } y \ge 7/8. \end{cases} \tag{19}$$

f and g are illustrated in Figure 4. Consider the set of points where f takes the value $(1,1)$, that is $\{\mathbf{x} \in [0,1]\times[0,1] : f(\mathbf{x}) = (1,1)\}$, and the corresponding set for g, $\{\mathbf{x} \in [0,1]\times[0,1] : g(\mathbf{x}) = (1,1)\}$. These sets are equal to $[1/2,1] \times [1/2,1]$ and $[0,1] \times [0,1/8] \bigcup [0,1] \times [7/8,1]$ respectively, and both have area $1/4$. Similarly the sets $\{\mathbf{x} : f(\mathbf{x}) = (0,0)\}$ and $\{\mathbf{x} : g(\mathbf{x}) = (0,0)\}$ have equal area (of $3/4$). As f and g take no other values, it follows that $\{\mathbf{x} : f(\mathbf{x}) \in B\}$ has the same area, or 2–dimensional Lebesgue measure, as $\{\mathbf{x} : g(\mathbf{x}) \in B\}$ for every Borel set $B \subset \mathbb{R}^2$. The restriction to Borel sets has the same purpose as in the previous section.

We will now use this property to give a formal definition of rearrangements for vector valued functions.

Definition Let Ω be a bounded region in $\mathbb{R}^n$, and let $f, g : \Omega \to \mathbb{R}^d$ be integrable functions. Then f and g are *rearrangements* if

$$\mu(\{\mathbf{x} : f(\mathbf{x}) \in B\}) = \mu(\{\mathbf{x} : g(\mathbf{x}) \in B\}) \tag{20}$$

for every Borel subset of $\mathbb{R}^d$, where μ denotes n–dimensional Lebesgue measure.

This definition is identical to that given in (16) for scalar valued rearrangements, except that d will now be greater than 1. There are several other definitions in the literature. Cullen, Norbury and Purser (1991) gave a different definition which is an extension of the characterisation of scalar valued rearrangements given above in (17). They required that for f and g as above, $\mu(\{\mathbf{x} : f(\mathbf{x}) \geq \mathbf{c}\}) = \mu(\{\mathbf{x} : g(\mathbf{x}\}) \geq \mathbf{c})$ for each $\mathbf{c} \in \mathbb{R}^d$, where the inequalities are calculated component by component. A third definition, used in Brenier (1991), requires that $\int_\Omega F(f)d\mathbf{x} = \int_\Omega F(g)d\mathbf{x}$ for each F in a subclass of the continuous functions from $\mathbb{R}^d$ to $\mathbb{R}$. We can unify these definitions as follows: (see Douglas (1996) for details); each may be reduced to the requirement that $\int_\Omega F(f)d\mathbf{x} = \int_\Omega F(g)d\mathbf{x}$ for each $F : \mathbb{R}^d \to \mathbb{R}$ in a class of functions $\mathcal{C}$. The different definitions above correspond to different choices of $\mathcal{C}$. $\mathcal{C}$ must be a sufficiently large class so that for each $n \in \mathbf{N}$ such that $\int_\Omega |f|^n$ is finite, we have $\int_\Omega |f|^n = \int_\Omega |g|^n$, i.e. all moments are preserved.

A vector valued function $f : \Omega \to \mathbb{R}^d$ may be written in terms of real valued components $f_i : \Omega \to \mathbb{R}$, where $f = (f_1, ..., f_d)$. If f and g are vector valued functions which are rearrangements then the corresponding components of f and g are rearrangements in the scalar valued sense. However it is important to note that the condition that each component of a vector valued function f is a rearrangement (in the scalar sense) of the corresponding component of a vector valued function g, is not sufficient for f to be a rearrangement of g in the vector valued sense. We give an example. Let f be as in the previous example, then

$$f_1(\mathbf{x}) = f_2(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in [1/2,1] \times [1/2,1], \\ 0 & \text{if } \mathbf{x} \notin [1/2,1] \times [1/2,1]. \end{cases} \tag{21}$$

Define

$$g(\mathbf{x}) = \begin{cases} (1,0) & \text{if } \mathbf{x} \in [1/2,1] \times [1/2,1], \\ (0,1) & \text{if } \mathbf{x} \in [0,1/2] \times [1/2,1], \\ (0,0) & \text{otherwise.} \end{cases} \tag{22}$$

Then $g_1 = f_1$ and

$$g_2(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in [0,1/2] \times [1/2,1], \\ 0 & \text{if } \mathbf{x} \notin [0,1/2] \times [1/2,1]. \end{cases} \tag{23}$$

It is easily seen that $f_1 \in R(g_1)$, and $f_2 \in R(g_2)$. However, $f \notin R(g)$, because f takes the value (1,1) on a region of area 1/4, while g never takes this value. Consequently in general we cannot apply scalar valued rearrangement results to the components of vector valued functions and hope to obtain results about vector valued rearrangements. Indeed, the theory of rearrangements of vector valued functions is less rich than that of rearrangements of scalar valued functions. We will see an example of this in the next section.

3.3 Rearrangements and trajectory mappings

When using rearrangement theory to study the properties of an evolving flow we consider the trajectory mapping that transforms initial positions $\mathbf{x}(\mathbf{a}, 0)$ of fluid particles $\mathbf{a}$ into positions $\mathbf{x}(\mathbf{a}, t)$ at a later time. We use the symbol s for this mapping. Any conserved particle property q is thus transported by this mapping, so that $q(\mathbf{x}(\mathbf{a}, t), t) = q(\mathbf{x}(\mathbf{a}, 0), 0)$. We write this more concisely as $q(t) = q \circ s$. If the fluid is incompressible it follows that, for any t, $q(\mathbf{x}, t)$ is a rearrangement of the initial data $q(\mathbf{x}, 0)$. This is because for any given subset U of the fluid at time t, for each particle in U we can find the original position of the particles at time zero and the set of these points has the same size as U. It follows that the trajectory mapping is a measure preserving mapping, the definition of which we give below.

Definition A *measure preserving mapping* from Ω to itself is a mapping $s : \Omega \to \Omega$ such that for each (measurable) set $U \subset \Omega$, $\mu(\{x : s(x) \in U\}) = \mu(U)$. We must again restrict our definition to sets where we can measure the size of the set - this excludes some pathological choices of U. Halmos (1974) shows that this is equivalent to requiring that for every integrable function f,

$$\int_\Omega f \circ s d\mu = \int_\Omega f d\mu. \tag{24}$$

If a measure preserving mapping s is one to one, and maps measurable sets to measurable sets, then the inverse of s exists and is also a measure preserving mapping. Such an s is called a *measure preserving transformation*. It can be shown that measure preserving mappings and transformations preserve (scalar or vector valued) rearrangements in the sense that if $f \in R(g)$ and s is a measure preserving mapping, then $f \circ s \in R(g \circ s)$. If s is a measure preserving transformation, then the reverse implication holds. Furthermore, $f \circ s \in R(f)$, and in particular, for a quantity σ which remains constant on fluid particles following the motion of the flow, $\sigma(t) \in R(\sigma(0))$.

Measure preserving mappings are also closely related to *canonical transformations* which are used in Hamiltonian descriptions of the flow. Sewell and Roulstone (1993) give a comparative study of several different definitions of canonical transformation, one of which is

Definition Given a transformation between two pairs of real scalars (x_1, x_2) and (X_1, X_2), with Jacobian

$$\det\left(\frac{\partial(X_1, X_2)}{\partial(x_1, x_2)}\right), \tag{25}$$

the transformation is *canonical* over a subspace of $\mathbb{R}^2$ if the Jacobian is equal to 1 over that subspace.

This definition requires the transformation to be differentiable. The definition of measure preserving mappings given above provides a natural extension to cases where the transformation is non-differentiable. Write $\mathbf{X} = (X_1, X_2)$ and $\mathbf{x} = (x_1, x_2)$. A co–ordinate transformation $\mathbf{X} = \mathbf{X}(\mathbf{x})$ could be considered *canonical* in a generalised sense for example if $\mathbf{X}$ is a measure preserving mapping which preserves important analytic structures such as the form of Hamilton's equations. Sewell and Roulstone review the qualitative properties of such transformations and applications in their paper.

4 Maximising or minimising functions with respect to rearrangements

As discussed in section 2, we are often interested in maximising or minimising functionals such as energy with respect to the set of rearrangements of a prescribed function. The most powerful results are available when there are inequalities which show that a special class of rearrangement achieves the minimisation or maximisation. In this section we review some important rearrangement inequalities; more on this topic can be found in Alvino, Lions and Trombetti (1989) and Burton (1987).

The first such special class are monotone rearrangements. Consider scalar rearrangements first. Let f, g be as in the first example of the section 3, that is

$$f(x) = x, \text{if } x \in [0,1], g(x) = \begin{cases} 1-2x & \text{if } x \in [0,1/2], \\ 2x-1 & \text{if } x \in [1/2,1]. \end{cases} \tag{26}$$

We have previously seen that $f \in R(g)$. f is an increasing function, and it can be proved that it is the only increasing function which is a rearrangement of g. More generally, it can be shown that for an integrable function f_0 defined on $[0,1]$, there is a unique element of $R(f_0)$ which is an increasing function. We call this element the *increasing rearrangement* of f_0, and denote it f_0^*. Increasing rearrangements are used in the following inequality. For scalar valued integrable functions f, g defined on the unit interval, we have

$$\int_0^1 fg \leq \int_0^1 f^* g^* \tag{27}$$

where f^*, g^* are the increasing rearrangements of f and g respectively. Indeed, if g has no level sets of non-zero length,

$$\int_0^1 fg^* < \int_0^1 f^* g^* \tag{28}$$

for every rearrangement f of f_0 which is not equal to f_0^*.

Any other rearrangement may be obtained from the increasing rearrangement by composition with a suitable measure preserving mapping. Ryff (1970) proved that, for $f \in L^1[0,1]$ i.e. for an integrable f defined on $[0,1]$, there exists a measure preserving mapping $s : [0,1] \to [0,1]$ such that $f = f^* \circ s$. Note that it is not always possible to get from one rearrangement to another rearrangement by a measure preserving mapping- for f and g as in the example above, no measure preserving mapping $s : [0,1] \to [0,1]$ such that $f = g \circ s$ exists.

There is an analogous concept for vector valued rearrangements. For $d \geq 2$, $\mathbb{R}^d$ is not well–ordered, therefore the concept of increasing does not make sense for vector valued functions. An increasing scalar valued function is the derivative of a convex function. Replacing derivative with gradient, this is a property which is well defined for vector valued functions, therefore we seek a rearrangement of a vector valued function equal to the gradient of a convex function. Let f be as in the vector valued examples of the previous section, that is $f : [0,1]^2 \to \mathbb{R}^2$ defined by

$$f(\mathbf{x}) = \begin{cases} (1,1) & \text{if } \mathbf{x} \in [1/2,1] \times [1/2,1], \\ (0,0) & \text{if } \mathbf{x} \notin [1/2,1] \times [1/2,1]. \end{cases} \tag{29}$$

Define $f^* : [0,1]^2 \to \mathbb{R}^2$ by

$$f^*(\mathbf{x} = (x,y)) = \begin{cases} (1,1) & \text{if } x + y \geq 1 - \frac{1}{\sqrt{2}}, \\ (0,0) & \text{otherwise.} \end{cases} \tag{30}$$

f^* is easily seen to be a rearrangement of f, and moreover $f^* = \nabla\Psi$, where $\Psi : [0,1]^2 \to \mathbb{R}$ is a convex function defined by

$$\Psi(\mathbf{x} = (x,y)) = \begin{cases} x + y - \left(1 - \frac{1}{\sqrt{2}}\right) & \text{if } x + y \geq 1 - \frac{1}{\sqrt{2}}, \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$

It can be shown that f^* is the unique rearrangement of f equal to the gradient of a convex function. More generally, if $\Omega \subset \mathbb{R}^d$ is a bounded closed convex set with smooth boundary, and $f_0 : \Omega \to \mathbb{R}^d$ is integrable, Brenier (1991) proved that there exists a unique $f_0^* \in R(f_0)$, with f_0^* equal to the gradient of a convex function. f_0^* is called the *monotone rearrangement* of f_0.

If f_0 satisfies a non-degeneracy condition, essentially that f_0 does not take a uniform value on any set of positive measure, Brenier (1991) has shown the existence of a unique measure preserving mapping $s : \Omega \to \Omega$ such that $f_0 = f_0^* \circ s$, and furthermore, s is the unique measure preserving mapping that maximises $\int_\Omega f_0(\mathbf{x}).s(\mathbf{x})d\mu(\mathbf{x})$. We can use this result to prove a rearrangement inequality. Let $f \in R(f_0)$ with $f \neq f_0^*$. Then there exists a measure preserving mapping s such that $f = f_0^* \circ s$. s is a measure preserving mapping, therefore we have

$$\int_\Omega f_0^*(\mathbf{x}).\mathbf{x}d\mu(\mathbf{x}) = \int_\Omega f_0^*(s(\mathbf{x})).s(\mathbf{x})d\mu(\mathbf{x}) \tag{32}$$

Using $f = f_0^* \circ s$ to substitute for f_0 on the right hand side of (32) gives

$$\int_\Omega f_0^*(\mathbf{x}).\mathbf{x}d\mu(\mathbf{x}) = \int_\Omega f(\mathbf{x}).s(\mathbf{x})d\mu(\mathbf{x}) \tag{33}$$

Applying the result of Brenier (1991) stated above, noting that the identity map is a measure preserving mapping from Ω to itself which is different from s, gives

$$\int_\Omega f_0^*(\mathbf{x}).\mathbf{x}d\mu(\mathbf{x}) > \int_\Omega f(\mathbf{x}).\mathbf{x}d\mu(\mathbf{x}) \tag{34}$$

Douglas (1996) showed that the above inequality still holds for f_0 which fail the non–degeneracy condition. However, Brenier (1991) shows that there is no inequality analogous to (27) for the monotone rearrangement of a vector valued function. This is an example where the theory of vector valued rearrangements is less rich than that of scalar valued rearrangements.

Another similar class of rearrangements are the symmetric rearrangements, used by Nycander (1995). His definition can be explained as

Definition Given a function of several variables $q(\mathbf{r})$ defined on a domain D, let $D_\mu = (\mathbf{r} \in D : q(\mathbf{r}) \geq \mu)$, and let D_μ^* be the sphere $r \leq \rho$ or $r < \rho$ with the same volume as D_μ. (This is called the symmetrised domain). Then, for any given $r \in D^*$, the symmetrically decreasing rearrangement of q is

$$q^*(\mathbf{r}) = \sup\{\mu : \mathbf{r} \in D_\mu^*\} \tag{35}$$

The symmetrized function is a decreasing function of $r = |\mathbf{r}|$.

We now illustrate the use of these results. As discussed in section 2, the solution of the semigeostrophic equations at any time can be considered as a minimum energy state, subject to simultaneous rearrangements of absolute momentum and potential temperature. The mathematical result on which this depends requires that the energy of the fluid can be minimised by rearranging a vector-valued function $\mathbf{X}$. The components of $\mathbf{X}$ correspond

to geostrophic and isentropic coordinates. The first result available of this type is the one stated in Cullen and Purser (1989):

Theorem 1. Let Ω be a connected convex region in real space with a smooth boundary. If the state of the fluid (at some fixed time) is described by values $\mathbf{X}_0 = (X_0, Y_0, Z_0)$ which are known on fluid particles, and the energy is defined by

$$E(\mathbf{X}) = \frac{1}{2}\int_\Omega X^2 + x^2 + Y^2 + y^2 d\mathbf{x} - \int_\Omega \mathbf{X}.\mathbf{x} d\mathbf{x} \tag{36}$$

for $\mathbf{X} \in \mathbf{R}(\mathbf{X}_0)$, where $\mathbf{x} = (x, y, z) \in \Omega$, then the fluid can be rearranged into a state $\mathbf{X}_0^* = \mathbf{X}_0^*(\mathbf{x})$ where

i) $E(\mathbf{X}_0^*) < E(\mathbf{X})$ for any rearrangement $\mathbf{X}$ other than $\mathbf{X}_0^*$ itself.

ii) $\mathbf{X}_0^* = \nabla\psi$ for some convex function ψ.

The solution can also be characterised in terms of the value ρ of an inverse potential vorticity $\frac{\partial \mathbf{x}}{\partial \mathbf{X}}$. Given $\rho = \rho(\mathbf{X})$ and provided that $\int \rho d\mathbf{X} = \mu(\Omega)$ (the total volume of Ω), the theorem allows the construction of a unique convex function $R(\mathbf{X})$ with $R = \mathbf{x}$ and $\frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \rho$, where ρ is a rearrangement of the given values.

We outline the proof of the theorem using rearrangement inequalities (a different approach was sketched by Cullen and Purser). The detailed proof is given in Douglas (1996). The first integral is conserved under vector valued rearrangements, therefore it is sufficient to show that $\int_\Omega \mathbf{X}.\mathbf{x} d\mu(\mathbf{x}) < \int_\Omega \mathbf{X}_0^*.\mathbf{x} d\mu(\mathbf{x})$ for every $\mathbf{X} \in \mathbf{R}(\mathbf{X}_0)$ with $\mathbf{X} \neq \mathbf{X}_0^*$. This follows by (33), even when there is a positive volume of fluid with a uniform value of $\mathbf{X}$.

Physically, fluid with uniform values of $\mathbf{X}$ can be rearranged without changing the energy, so that we can expect non-uniqueness of the minimising rearrangement in terms of the particle positions. However the values $\mathbf{X}(\mathbf{x})$ are still unique.

Nycander (1995) uses a similar strategy to prove that the energy of a vortex patch in a background shear with uniform vorticity can be maximised with respect to rearrangements. The proof uses an inequality like (27) to show that the energy is maximised by a symmetric rearrangement as defined in (35). He has to be careful to prove that the maximum is actually attained by a rearrangement.

Burton (1987) studied variational problems for vortex rings in an ideal fluid. He used an inequality similar to (27) in the proof that a functional, representing kinetic energy and momentum, is maximised relative to the set of rearrangements of a given function, where the given function is a "vorticity–like" quantity. He used another rearrangement inequality to show that a maximiser yields an associated stream function which satisfies the equation of a steady flow.

5 Limits of sequences of rearrangements

Rearrangement inequalities can only be used to study minimisation or maximisation of integrals in special cases. We also need methods which can be used more widely. A standard approach would be to construct a minimising or maximising sequence of rearrangements, and show that this sequence, or a subsequence, has a limit which is also a rearrangement and which achieves the required extreme value. This is the approach to energy minimisation used in Cullen and Purser (1989). However such a limit may not exist. A particular difficulty is that the set of rearrangements of a given function includes arbitrarily fine-grained rearrangements. We must then decide whether to include the limit state in which the fluid is allowed to mix. A similar problem arises in studying the long time behaviour of a conserved quantity like potential vorticity which is rearrangement preserved. Fine grained rearrangements of potential vorticity and tracer concentration are observed at the stratospheric vortex edge. Therefore we wish to review some properties of sets of rearrangements.

The most useful set of functions to work with for meteorological data is the set of square integrable functions $L^2(\Omega)$. Functions in this space have a finite 'energy', but can have arbitrarily small scale oscillations. This is typical, for instance, of atmospheric wind fields. Formally, $f \in L^2(\Omega)$ if $\int_\Omega f^2 d\mu < \infty$, and we can define a norm on $L^2(\Omega)$ by $||f||_2 = \{\int_\Omega f^2 d\mu\}^{\frac{1}{2}}$.

We now recall some standard definitions, which will enable us to say whether a sequence of functions converges to a limit. For more detail see Friedman (1982), sections 3.2, 4.10, 4.14.

Definitions A sequence $\{f_n\} \subset L^2(\Omega)$ converges strongly to $f \in L^2(\Omega)$ if $||f_n - f||_2 \to 0$

as $n \to \infty$. $S \subset L^2(\Omega)$ is *strongly closed* if for every sequence $\{f_n\}$ in S which converges strongly to a $f \in L^2(\Omega)$, say, then the limit function f is actually in S.

A set $S \subset L^2(\Omega)$ is *strongly compact* if every sequence in S has a subsequence which converges strongly to an element of S. A strongly compact set is strongly closed.

It is useful to have another notion of convergence and compactness which can "ignore" small scale oscillations, and thus be more realistically applied to meteorological data. We give an example. For each $n \in \mathbb{N}$, let $f_n \in L^2([0, 2\pi])$ be defined by $f_n(x) = \sin nx$. The sequence $\{f_n\}$ does not converge in $L^2([0, 2\pi])$, nor does any subsequence. However the f_n approach the constant function with value 0 in an averaged sense, as $\int_0^{2\pi} f_n g \to 0$ as $n \to \infty$ for any $g \in L^2(\Omega)$. We say that f_n *converges weakly* to 0. We give the formal definition.

Definition For all positive integers n, let $f_n \in L^2(\Omega)$, and further suppose that $f \in L^2(\Omega)$. We say that f_n *converges weakly* to f if $\int_\Omega f_n g \to \int_\Omega fg$ as $n \to \infty$ for every $g \in L^2(\Omega)$. Strong convergence implies weak convergence, but not vice versa.

A set $S \subset L^2(\Omega)$ is *weakly (sequentially) compact* if every sequence in S has a subsequence converging weakly to an element of S. For any sequence in S which has a weak limit, the weak limit belongs to S. Strong compactness implies weak (sequential) compactness.

The set of rearrangements of a non–negative function $f_0 \in L^2(\Omega)$ is (strongly) closed, but not strongly compact or weakly compact unless f_0 is a constant function. For example, there are sequences of rearrangements which converge weakly to limits which are not rearrangements of the original function (from which it follows that the set is not weakly

compact). Let $f_0 : [0,1] \to \mathbb{R}$ be defined by

$$f_0(x) = \begin{cases} 0 & \text{if } x \in [0, 1/2], \\ 1 & \text{if } x \in [1/2, 1]. \end{cases} \tag{37}$$

Define, for $n \in \mathbb{N}$,

$$f_n(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0 & \text{if } x \in (m/n, (2m+1)/2n], \\ 1 & \text{if } x \in [(2m+1)/2n, (m+1)/n]. \end{cases} \tag{38}$$

where $m = 0, 1, ..., n-1$. The functions f_3 and f_8 are illustrated in Figure 5. For each $n \in \mathbb{N}$, f_n is equal to zero on a set of length $1/2$, and equal to 1 on a set of length $1/2$, therefore f_n is a rearrangement of f_0 for each $n \in \mathbb{N}$. However, given any $g \in L^2(0,1)$, it may be shown that $\int_\Omega f_n g d\mu \to 1/2 \int_\Omega g d\mu$ as $n \to \infty$, that is f_n converges weakly to the constant function with value $1/2$, which is not a rearrangement of f_0.

This behaviour can make it difficult to maximise or minimise a functional with respect to rearrangements. It is quite possible that there will be a sequence of rearrangements giving progressively less energy, for instance, but the limit function will not be a rearrangement. To overcome this difficulty, we work with the weak closure of the set of rearrangements of a given function, the smallest weakly compact set that contains the set of rearrangements. This set contains all the weak limits of sequences of rearrangements, and may therefore, for instance, allow us to construct the functions that maximise or minimise energy. Given a non–negative function $f_0 \in L^2(\Omega)$, this set is equal to the closed convex hull of the set of rearrangements. Douglas (1994) gave the following characterisation

$$\overline{\text{conv}R(f_0)} \;=\; \{f \;\text{ non}-\text{negative} \mid \int_\Omega (f-\alpha)_+ d\mu \le \int_\Omega (f_0-\alpha)_+ d\mu$$

$$\text{for each } \alpha > 0, \int_\Omega f d\mu = \int_\Omega f_0 d\mu.\} \tag{39}$$

where the + subscript denotes taking the positive part of the function. If we choose f_0 as in the previous example, that is $f_0 : [0,1] \to I\!R$ is defined by

$$f_0(x) = \begin{cases} 0 & \text{if } x \in [0, 1/2], \\ 1 & \text{if } x \in [1/2, 1], \end{cases} \tag{40}$$

it can be shown that any integrable function $\varphi : [0,1] \to I\!R$ satisfying $0 \leq \varphi(x) \leq 1$ for each $x \in [0,1]$, and $\int_0^1 \varphi d\mu = 1/2$, belongs to $\overline{\text{conv } R(f_0)}$. This illustrates that $\overline{\text{conv } R(f_0)}$ may be a large class of functions, in particular it includes the constant value $1/2$ which is certainly not a rearrangement of f_0. In general, all rearrangements are included, as are functions derived by smoothing a rearrangement, if the smoothing preserves the value of the integral, and does not introduce new extreme values. Thus the limit of a sequence of fine grained rearrangements will be a smoothed 'average' function. Physically, including these limit functions can be thought of as allowing for a small but finite viscosity or conductivity.

We illustrate these concepts by considering two energy minimisation problems discussed in section 2. We use the simple one-dimensional examples of Shutts *et al* (1988). Suppose we minimise

$$\frac{1}{2}\int_0^1 v_g^2 dx = \frac{1}{2}\int_0^1 (x - X)^2 dx \tag{41}$$

over $X \in R(f_0)$, where f_0 is as in (37). $\int_0^1 X^2 dx$ has the same value for each $X \in R(f_0)$, therefore minimising (41) is equivalent to maximising $\int_0^1 xX dx$ over $X \in R(f_0)$. This integral is maximised by taking $X = f_0^* = f_0$, where f_0^* is as defined by (26). This choice

yields a minimum energy of 1/24 for the integral (41). However if we minimise (41) with respect to X in the weak closure of the set of rearrangements of f_0, the function $X(x) = x$ belongs to this set, therefore we can attain a zero minimum energy state. Note that $X(x) = x$ is the weak limit of a sequence of rearrangements. Therefore a lower energy can be achieved by minimising relative to the weak closure of the set of rearrangements, rather than by minimising relative to the set of rearrangements itself.

Now consider minimising the energy of a vertical column of fluid. Physically, the solution is a rearrangement with the potential temperature monotonically increasing with height, corresponding to the inequality quoted in section 4. The energy can be written, using the pseudo-height coordinate of section 2(iii), as

$$\int -\frac{g\theta z}{\theta_0} \tag{42}$$

Minimising (42) is equivalent to maximising $\int zZ$, where Z is the isentropic coordinate $\frac{g\theta}{\theta_0}$. Suppose that we maximise $\int_0^1 zZ$ over $Z \in R(f_0)$, where f_0 is as in (37). The maximum is attained by choosing $Z = f_0^* = f_0$, with $\int_0^1 zf_0 = 3/8$. If we maximise $\int_0^1 zZ$ over Z belonging to the weak closure of the set of rearrangements of f_0, which includes, for example, the constant function 1/2, no larger value can be obtained. Therefore we cannot achieve a lower energy by maximising relative to the weak closure of the set of rearrangements in this example.

The difference between these examples follows from the form of the energy functionals. In the case of (41), $\int X^2 dx$ is conserved under rearrangements of X, but is not necessarily conserved when the weak limit is taken. For example, suppose we have a sequence $\{f_n\}$ of rearrangements of f_0 which converges weakly to the identity function id. For each n, $\int_0^1 f_n^2 = 1/2$, but $\int_0^1 id^2 = 1/3$. On the other hand, if $\{X_n\}$ converges weakly to X, then $\int xX_n dx \to \int xX_n dx$ as $n \to \infty$.

In the physical case, we have to consider whether terms in functionals we are extremising are robust against smoothing. The distinction is reflected in observations of the atmosphere. A situation where the potential temperature is layered in the vertical is quite common, and reflects that a lower energy cannot be reached by mixing the fluid. The horizontal equivalent of layering of absolute momentum $(v + fx)$ is not as normally observed, since a lower energy can be reached by mixing the fluid to create more uniform values of v.

However, if a state where

$$v + fx = f_0 \tag{43}$$

is created, with f_0 defined as in (37), the mixing to attain uniform v is difficult to achieve. We can construct a sequence of rearrangements of f_0 which converge weakly to the well mixed solution. However each element of the sequence has larger energy than the state described by f_0, therefore we cannot reach the well mixed solution in this way. Instead, the uniform v would have to be reached by slow 'diffusion' of the v profile.

6 Time evolution of rearrangements of vorticity or potential vorticity

In this section we discuss constraints on flow evolution resulting from the Lagrangian conservation of vorticity or potential vorticity. The two types of result considered are the behaviour of the long time evolution of the flow, in particular whether there is an enstrophy cascade; and the characterisation and stability of steady states. The case of equations (3) and (4) has been extensively studied, and we review some key results from the literature which make explicit or implicit use of rearrangement theory. Most of these results carry over to the quasi-geostrophic case (5). The semi-geostrophic case is a little different, because the Monge-Ampere equation rather than Poisson's equation is used in the potential vorticity inversion. It is shown that this inhibits the enstrophy cascade, but prevents global stability results being obtained for steady state solutions. We interpret the evolution equations as rearranging vorticity or potential vorticity, which allows us to consider possible non-smooth solutions.

6.1 Barotropic vorticity equation

It is possible that increasingly fine-grained solutions will be developed from smooth initial data through time evolution, for instance the 'enstrophy cascade' described by Vallis (1992). As discussed in Section 5, the set of rearrangements is not compact, therefore there exist sequences of rearrangements such that no subsequence converges to a rearrangement. One way of overcoming this difficulty is to work with the weak closure of the set of rearrangements, which we defined in the previous section. We examine this idea in more detail in the next two paragraphs. An alternative strategy is to identify a "reachable" compact subset of the set of rearrangements - rearrangements that can actually be reached in the

time evolution. One way to do the latter is to bound the gradients of the vorticity or potential vorticity in terms of the initial gradients. In the case of the barotropic vorticity equation (3,4), the vorticity gradients obey the equation

$$\frac{D}{Dt}\nabla(\zeta+f)+\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x}\\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}\end{pmatrix}\nabla(\zeta+f)=0 \tag{44}$$

This can be written in terms of the streamfunction ψ as

$$\frac{D}{Dt}\nabla(\zeta+f)+\begin{pmatrix}-\frac{\partial^2\psi}{\partial x\partial y} & \frac{\partial^2\psi}{\partial x^2}\\ -\frac{\partial^2\psi}{\partial y^2} & \frac{\partial^2\psi}{\partial y\partial x}\end{pmatrix}\nabla(\zeta+f)=0 \tag{45}$$

These equations can be used to estimate the rate of increase of vorticity gradients. The derivatives of the streamfunction can be bounded because the vorticity is bounded by its initial values and by using the boundary condition specified in (4). This is the basis of the results of Kato (1967) and Bennett and Kloeden (1981). The fact that the subset of rearrangements of $\zeta+f$ with bounded gradients is compact makes the proof possible. The size of the bound is very large because the most pessimistic possible assumption has to be used. As stated by Bennett and Kloeden, the magnitude of the deformation field has to be assumed everywhere equal to its maximum value, the field everywhere aligned with the direction of the vorticity gradient, and the magnitude of the vorticity gradient everywhere equal to its maximum. This type of result is therefore of limited practical use. More information about the flow geometry must be used to restrict the possible rearrangements.

The second method is to embed the set of rearrangements in a larger set, the weak closure of the set of rearrangements. In section 5 it was shown that this set includes all

smoothings of the initial vorticity distribution, as well as the rearrangements themselves. This set is weakly compact. There is no control on the vorticity gradients, so the solution is 'generalised', and can include 'fractal' vorticity distributions. This allows us to consider arbitrarily fine-grained solutions resulting from an enstrophy cascade, and the inclusion of the smooth limit functions means that we can also represent the effect of a small but undefined viscosity.

Once this is done, standard methods can be used to show that a solution exists. The technique is as follows. Given an initial distribution of (potential) vorticity q_0 on a region Ω, use the above discussion to identify a set $E(q_0)$, a 'reachable' subset of the weak closure of the set of rearrangements of q_0, which contains all rearrangements of q_0 which can be reached by evolution of the initial data according to equation (1), together with all the limit points of this set. The flow geometry ensures that this set is (strongly) compact and convex. We seek a solution over an arbitrary time interval $[0, T]$. Make a first guess to this solution by setting q at each time t to be an arbitrary reachable rearrangement $q_0(t)$ of the initial data. By construction, this guess is within $E(q_0)$. At each time, solve the equivalent of (2) to give a velocity field $u_0(t)$:

$$H(u_0) = q_0 \tag{46}$$

Use this velocity field to transport the initial data, according to equation (1). The result is a second 'guess' $q_1(t)$ to the solution. q_1 is a rearrangement of q_0, contained in the compact and convex set $E(q_0)$. The mapping of the first guess q_0 to q_1 is shown to be a continuous function from $E(q_0)$ to $E(q_0)$. It is then possible to use a standard mathematical theorem to show that this mapping has a fixed point q_f. By construction, this is a solution of (1) and (2).

These results on their own do not give any reason why the vorticity distribution should not gradually cascade into fine grained chaos. Such a cascade is the basis of standard two-dimensional turbulence theory, and is characterised by an energy spectrum obeying a k^{-3} power law. However, many computations, for example Ehlmaidi et al. (1993), illustrate that the flow is not chaotic everywhere. There are coherent regions where the streamlines are closed. On the edges of these are regions where there is rapid straining motion. There are also regions of background turbulence with no clear structure. Weiss (1991) classifies these regions by the sign of the determinant $\psi_{xy}^2 - \psi_{xx}\psi_{yy}$ of the matrix appearing in equation (45) which governs the rate of growth of the vorticity gradient. An identical form of equation governs the growth of line elements in the flow. If this matrix varies slowly following the motion, the eigenvalues will determine whether the gradient can increase. If the determinant is negative, then both eigenvalues are pure imaginary and the maximum growth is linear in time, see Jordan and Smith (1977). If we allow for general time dependence of the matrix coefficients there may still be exponential growth due to resonant effects. If the determinant is positive there is exponential growth. The coherent regions correspond to a negative determinant, the rapid straining regions to a positive determinant, and the rest of the region to an indeterminant mixture of signs. It is then necessary to understand how this pattern is maintained.

Dritschel (1988a) quotes and proves some theorems which limit the growth of perturbations to plane parallel, axisymmetric or spherically symmetric flow using the barotropic vorticity and quasi-geostrophic equations. He shows that, by using the rearrangement property that the area enclosed within each vortex contour is conserved, stronger results can be proved than were previously available. In particular it is shown that if the vorticity

gradient is monotonic

i) The integrated magnitude of the areal displacement of a vortex contour from its equilibrium position is conserved.

ii) The mean square displacement of a vortex contour normal to itself is conserved in time.

However, it is also shown by Dritschel (1988b) that, within these constraints, vortex contours can exhibit 'repeated filamentation', leading after a long time to a highly complex boundary shape. Constantin and Bertozzi (1993) have proved that a line element of a boundary separating two different values of the vorticity can stretch at a rate proportional to $exp(e^t)$. This an example of a rearrangement which is highly chaotic and indistinguishable in the limit from an 'averaged' solution. However, results (i) and (ii) above show that this behaviour will not alter the gross shape of a circular vortex patch. These results are proved using information about the geometry of the flow associated with the vortex patch and are consistent with the picture from the computations described above. The vorticity can thus only evolve within a restricted class of rearrangements, which all share the same large scale structure.

Dritschel's (1988a) results depend on bounding a norm of the displacement of vorticity contours under the constraint of energy conservation. These bounds are much sharper, and a vortex patch more stable, if the patch corresponds closely to a maximum or minimum of the energy subject to rearrangements of the vorticity. In such a case only small displacements of the contours can be consistent with energy conservation. Nycander(1995) establishes conditions for an energy maximum. He shows that, given a uniform shear flow, and an isolated region of (non-uniform) vorticity, this region can be rearranged to give an energy maximum. In the case of a patch of constant vorticity, the maximum energy state

is an elliptic patch. In more general cases the shape cannot be determined analytically.

6.2 Semi-geostrophic equations on the f plane

The evolution equations (12) take the form

$$\begin{aligned} \dot{X} &= f(\frac{\partial R}{\partial Y} - Y) \\ \dot{Y} &= X - f(\frac{\partial R}{\partial X}) \\ \dot{Z} &= 0 \end{aligned} \tag{47}$$

It was shown in section 2 that R is a convex function. The equivalent equation to (45) is

$$\frac{D}{Dt}\nabla\rho + \begin{pmatrix} -\frac{\partial^2\Psi}{\partial X\partial Y} & \frac{\partial^2\Psi}{\partial X^2} \\ -\frac{\partial^2\Psi}{\partial Y^2} & \frac{\partial^2\Psi}{\partial Y\partial X} \end{pmatrix} \nabla\rho = 0 \tag{48}$$

where

$$\Psi = \frac{1}{2}(X^2 + Y^2) - R \tag{49}$$

The significant difference between this equation and (45) is that Ψ is the difference between a convex function and a solid body rotation represented by the streamfunction $\frac{1}{2}(X^2+Y^2)$. The latter has no topological effect on the flow. As discussed above, this suggests that, if the matrix coefficients do not vary along trajectories, the potential vorticity gradients and line elements can grow at most linearly in time. The only possible exponential growth is due to resonant instabilities because of time dependence of the matrix, e.g. Jordan and Smith (1977), pp.237-242. It is not clear if these resonances are possible in this type of flow.

Even if the maximum growth allowed is linear, however, the growth permitted in practice is still likely to be quite fast when such effects as initial filamentation of the stratospheric polar vortex are studied. This is because the solid body rotation has the inertial period. Thus the actual evolution will look quite complex, as it must if it is to be relevant to atmospheric flow, though it has very simple topological behaviour. The importance of the different dynamics comes in restricting filamentation and mixing at smaller scales. Formally, the flow will be non-chaotic in a Lagrangian sense, Babiano et al (1994). This means that initially close particles cannot separate exponentially fast, so that chaotic 'stirring' and mixing of air masses is prevented. Note that Babiano et al. point out that Lagrangian and Eulerian predictability are different properties, and non-chaotic Lagrangian dynamics can still be chaotic in an Eulerian sense, with large sensitivities to small changes in the initial data.

As in the case of the barotropic vorticity equation, we can also seek to use the rearrangement property to characterise steady state solutions and their stability. We investigate this for the barotropic semi-geostrophic problem. Kushner and Shepherd (1995) studied this by extremising the energy with respect to arbitrary perturbations of ρ. However, in that formulation the energy is non-quadratic in the potential vorticity perturbation and they were only able to obtain small amplitude results. We show that stronger results can be obtained by restricting the perturbations to rearrangements of ρ. This is an example of a more general procedure for extremisation of integrals respecting dynamical constraints (Morrison (1993)).

A rearrangement perturbation to ρ is defined by keeping ρ fixed on particles in $\mathbf{X}$ space and perturbing $\mathbf{X}$. This perturbation must be non-divergent, so can be written as

$(-\frac{\partial \alpha}{\partial Y}, \frac{\partial \alpha}{\partial X})$ for an arbitrary function $\alpha(X, Y)$.

We show first that, as in the barotropic case, a steady state of the evolution equations corresponds to a stationary point of the energy under rearrangements of ρ. The perturbation to E can then be written

$$\delta E = \int_{(X,Y)} ((X-x)\delta X + (Y-y)\delta Y - X\delta x - Y\delta y)\rho d\mathbf{X} \tag{50}$$

where the integration is taken over particles, so that there is no $\delta\rho$ and we have used the invariance of $\int_\Omega x^2 + y^2 d\mathbf{x}$. The results of section 4 show that

$$\int_{(X,Y)} (-X\delta x - Y\delta y)\rho d\mathbf{X} = 0 \tag{51}$$

because this is the variation of $\mathbf{x}$ with $\mathbf{X}$ fixed considered in Theorem 1. Substituting the definitions of Ψ and $\delta\mathbf{X}$ gives

$$\int_{(X,Y)} (-\frac{\partial \Psi}{\partial X}\frac{\partial \alpha}{\partial Y} + \frac{\partial \Psi}{\partial Y}\frac{\partial \alpha}{\partial X})\rho d\mathbf{X} = 0 \tag{52}$$

so

$$\int_{(X,Y)} (\nabla.(\alpha\rho\frac{\partial \Psi}{\partial Y}, -\alpha\rho\frac{\partial \Psi}{\partial X}) - \alpha(-\frac{\partial \Psi}{\partial X}\frac{\partial \rho}{\partial Y} + \frac{\partial \Psi}{\partial Y}\frac{\partial \rho}{\partial X}))d\mathbf{X} = 0 \tag{53}$$

Assuming that ρ vanishes at a sufficiently large $|\mathbf{X}|$ and requiring (53) to hold for arbitrary α gives

$$-\frac{\partial \Psi}{\partial X}\frac{\partial \rho}{\partial Y} + \frac{\partial \Psi}{\partial Y}\frac{\partial \rho}{\partial X} = 0 \tag{54}$$

This condition is precisely that for the flow to be steady as we can see from an Eulerian form of (47).

We then seek to identify stable steady states. Following Nycander, these will be maxima or minima of the energy subject to rearrangements of ρ because a flow initiated by a small perturbation to the steady state will have to conserve energy and so be trapped in the neighbourhood of the steady state. First note that the global energy extremisation methods used by Nycander do not give useful information in the semigeostrophic case. The energy for the barotropic problem can be derived from (36) as

$$\begin{aligned} E(\mathbf{X}) = & \quad \tfrac{1}{2}\int_\Omega (X-x)^2 + (Y-y)^2 d\mu(\mathbf{x}) \\ = & \quad \tfrac{1}{2}\int_{(X,Y)} \left((X-x)^2 + (Y-y)^2\right) \rho d\mathbf{X} \end{aligned} \tag{55}$$

Consider a rectangular region Ω in the (x, y) plane. Consider the special case where ρ is non-zero over a region Γ in (X, Y) space with the same area as Ω. Then the average value of ρ is 1. Using a two-dimensional version of the construction in (38) , we can rearrange ρ so that its average value is 1 over every arbitrarily small subregion of Γ, where Γ is defined by the same coordinates in (X, Y) as Ω is in (x, y). Then the weak limit of these mappings from $\mathbf{X}$ to $\mathbf{x}$ is clearly the identity and the energy of the weak limit is zero. Similarly, because there is no restriction on the position of Γ in (X, Y) space under rearrangements, an arbitrarily large energy can be reached by moving Γ to a large $|\mathbf{X}|$. This is true even if we apply the constraints from the evolution equations (47) that the centroid of Γ is fixed, and that Γ remains connected. This situation does not arise in the barotropic or quasi-geostrophic case because vorticity is there being rearranged in physical space, rather then an unbounded phase space. Kushner and Shepherd identified this point when considering the Charney-Stern theorem for straight semigeostrophic flow. The free boundary effectively makes all states unstable.

Useful results have to be obtained by seeking local extrema under rearrangements. In order to find a suitable problem, it is best to start with a geometrical picture of the solution in physical space. Given a region Ω, a no-flow solution is represented by a the convex surface $P = \frac{1}{2}(x^2 + y^2)$, so that $\mathbf{x} = \mathbf{X}$, Γ has the same coordinates as Ω and $R = \frac{1}{2}(X^2 + Y^2)$. The solution corresponding to a localised vortex 'patch' is obtained by concentrating the curvature of part of the surface P, with compensating reduction of the curvature in the surrounding region. A one-dimensional cross-section of this is shown in Fig.6. Two independent patches P_1 and P_2 are illustrated. The perturbations to the curvature have to be consistent with the overall convexity of P. Under this construction we can be sure that $\mathbf{x} - \mathbf{X} = 0$ away from the patch, so that any motions generated by perturbations to the patch will be locally confined and in particular will not be able to exchange energy with perturbations to the boundary of Γ. In effect, the long range solution is shielded from the local perturbation to ρ by the surrounding reverse perturbation.

We seek steady solutions consisting of a region of uniform $\rho = \rho_1$ surrounded by a region of uniform $\rho = \rho_2$, with areas chosen so that $\int \rho = 1$. Uniformity of ρ requires

$$\frac{\partial^2 R}{\partial X^2}\frac{\partial^2 R}{\partial Y^2} - (\frac{\partial^2 R}{\partial X \partial Y})^2 = \text{constant} \tag{56}$$

A particular solution is an elliptical patch with $\rho = \rho_1$ surrounded by an elliptical annulus with $\rho = \rho_2$. Without loss of generality assume that the (X, Y) axes are the principal axes of the ellipses and the eccentricity of the ellipses is e^2. Then the inner ellipse is given by

$$\pi(X^2/e + Y^2 e) = A_1 \tag{57}$$

and the outer ellipse by

$$\pi(X^2/e + Y^2 e) = (A_1 + A_2) \tag{58}$$

where A_1 and A_2 are the areas of the two regions of uniform ρ. The streamfunction is defined by

$$\Psi = \begin{cases} a_1(X^2/e + Y^2 e) & \text{if } (X^2/e + Y^2 e) \leq A_1/\pi, \\ (a_2/\pi)(A_1 + A_2) - a_2(X^2/e + Y^2 e) & \text{if } A_1/\pi \leq (X^2/e + Y^2 e) \leq (A_1 + A_2)/\pi. \end{cases}$$

where continuity of Ψ on the inner ellipse requires

$$(a_1 + a_2)A_1 = a_2(A_1 + A_2) \tag{59}$$

and

$$\rho_1 = 1 + 2a_1(e + 1/e) + 4a_1^2 \tag{60}$$

with a similar equation for ρ_2 and a_2. The total energy is given by the integral of

$$4(a_1/e)^2 X^2 + 4(a_1 e)^2 Y^2 \tag{61}$$

over the inner ellipse and

$$4(a_2/e)^2 X^2 + 4(a_2 e)^2 Y^2 \tag{62}$$

over the elliptical annulus. After some algebra, and the substitution $\sin\theta = X\sqrt{(\frac{\pi}{A_1 e})}$, the energy integrated over the inner ellipse is

$$8a_1^2(\frac{A_1}{\pi}^2 \int_{-\pi/2}^{\pi/2} (\sin^2\theta\cos^2\theta/e + \frac{1}{3}e\cos^4\theta)d\theta \tag{63}$$

These two terms can be shown to be equal if $e = 1$. Further algebra shows that the total integral over both regions has the same form, with the coefficient outside the integral replaced by

$$\frac{8}{\pi^2}(A_1^2(a_1^2 - a_2^2)) + 8a_2^2(A_1 + A_2)^2 \tag{64}$$

The energy therefore tends to infinity as e tends to zero or infinity, corresponding to a straight filament of anomalous ρ, and is minimised for $e = 1$, corresponding to a circular region.

Use of Nycander's arguments suggest that the circular perturbed region is stable. However, this is only the case if rearrangements approximating mixing are excluded, since it is clear that the energy would be reduced towards zero if the anomalous regions were mixed. This restriction could be justified if the rearrangements were restricted to displacements generated by a perturbation to R which maintained its convexity, since actual evolution under (47) is so restricted. Even then, it is necessary to check that no lower energy can be reached, since our analysis is restricted to elliptic patches, unlike Nycander's which can use rearrangement inequalities because of the special form of the energy integral.

The straight flow limit corresponding to a maximum energy can only lead to a useful stable steady state in the case of zonal flows with zonally periodic boundary conditions. Otherwise the perturbed region must interact with the boundary of Γ and boundary displacements cannot be justifiably excluded. Supposing the periodicity to be in the X direction, a greater energy can be achieved by arranging regions with values $\rho = \rho_2, 1, \rho_1$ consecutively in the Y direction, thus maximising $(y - Y)$. This monotonic arrangement corresponds to the Charney-Stern theorem, but is still not stable to perturbations in the Y boundaries of Γ. The case of the localised jet, with central value ρ_1 surrounded by strips of value ρ_2 may still be stable to local perturbations. Our result suggests that wavelike perturbations to any of the interfaces between values of ρ would reduce the energy since they approximate 'mixing' rearrangements which will certainly reduce the energy. It is possible that stability could be recovered by restricting the perturbations using the convexity of R as above. However, this needs to be fully analysed.

The conclusions which can be drawn from this case are *(i)* Useful results are only likely to be obtained by restricting the class of perturbations to those which arise naturally in the flow evolution. *(ii)* Straight and circular patches of uniform ρ are the only likely candidates for stable steady states.

7 Discussion

We have discussed a number of applications of Lagrangian mathematics, particularly the use of rearrangements arising from an area or volume preserving map. We have shown that in various balanced models described in terms of potential vorticity and an invertibility principle that useful nonlinear and large amplitude information can be obtained about the flow evolution. This includes an important identification of flows where the potential enstrophy cascade can be prevented, and variational arguments to characterise steady states. We have shown that the most powerful results occur in cases where we can use a rearrangement inequality to extremise a function over all possible rearrangements. Examples quoted used monotone or symmetrising rearrangements. It is important to identify more such situations and exploit them.

Other possible applications are to numerical methods and data assimilation. It is clearly desirable to satisfy the rearrangement property in a numerical solution of the evolution equation for potential vorticity, and to preserve it as well as possible in formulations where potential vorticity is not used as a model variable. This should form part of the assessment of competing numerical methods. If the non-chaotic dynamics associated with advection by a convex streamfunction is important, then a property equivalent to this has to be preserved in the numerical approximation, so that the enstrophy cascade is absent there also and no enstrophy sink has to be provided by adding artificial viscosity. In data assimilation, it may be appealing to try and work with displacements or rearrangements of fields in the analysis procedure, rather than with Eulerian perturbations. This could include minimisation over rearrangements, possibly restricted to those consistent with convexity conditions, as part of a general variational framework.

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8 Figure Captions

Figure 1. Graphs of $f(x) = x, x \in [0,1]$ and

$$g(x) = \begin{cases} 1-2x & \text{if } x \in [0,1/2], \\ 2x-1 & \text{if } x \in [1/2,1]. \end{cases} \tag{65}$$

illustrating the reason why f is a rearrangement of g.

Figure 2. Graphs of the inverse images of f and g, with $f = f(x)$, $g = g(x)$ defined as in Fig.1. The sets $f^{-1}[1/2,1]$ and $g^{-1}[1/2,1]$ are illustrated.

Figure 3. Graphs of two different rearrangements of $f(x) = x, x \in [0,1]$. The shaded areas are the same.

Figure 4. Values of the vector valued functions $f = f(x,y)$ and $g = g(x,y)$ defined on $[0,1] \times [0,1]$ by (18) and (19) respectively.

Figure 5. The rearrangements $f_3 = f_3(x)$ and $f_8 = f_8(x)$ defined by equation (38) of the function $f_0 = f_0(x)$ defined by equation (37).

Figure 6. The effect on the surface $P = P(x)$ of local concentrations of its curvature.

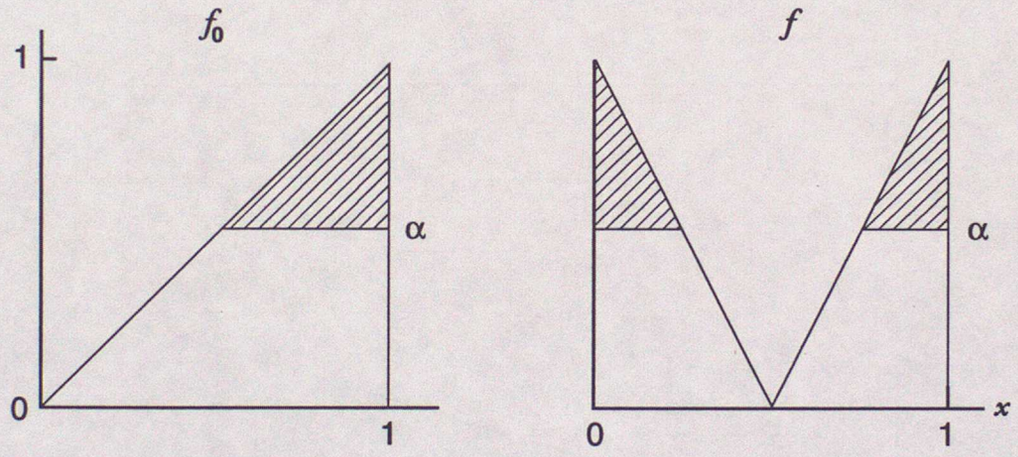
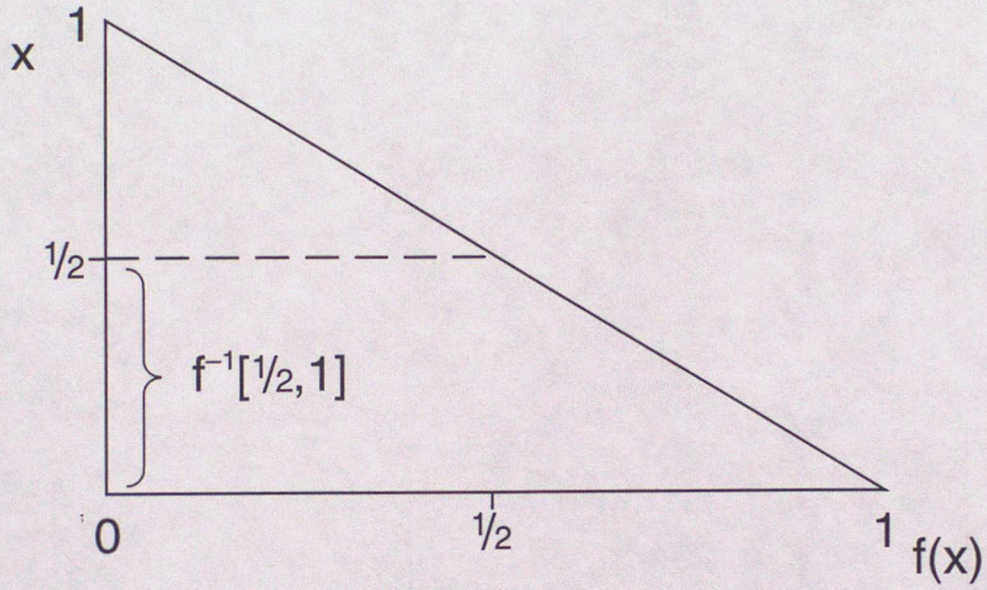


Fig.1



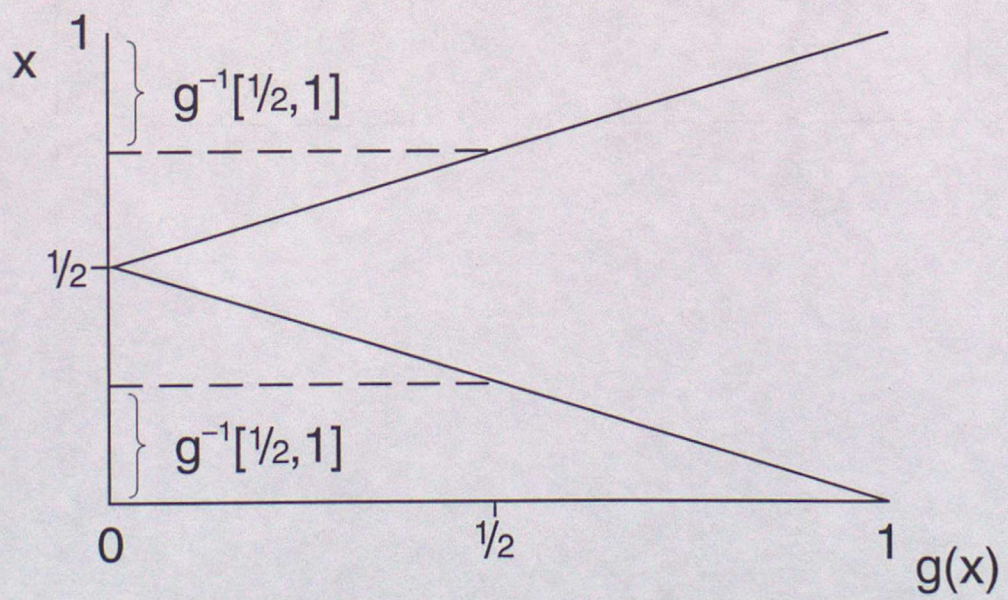


Fig.2

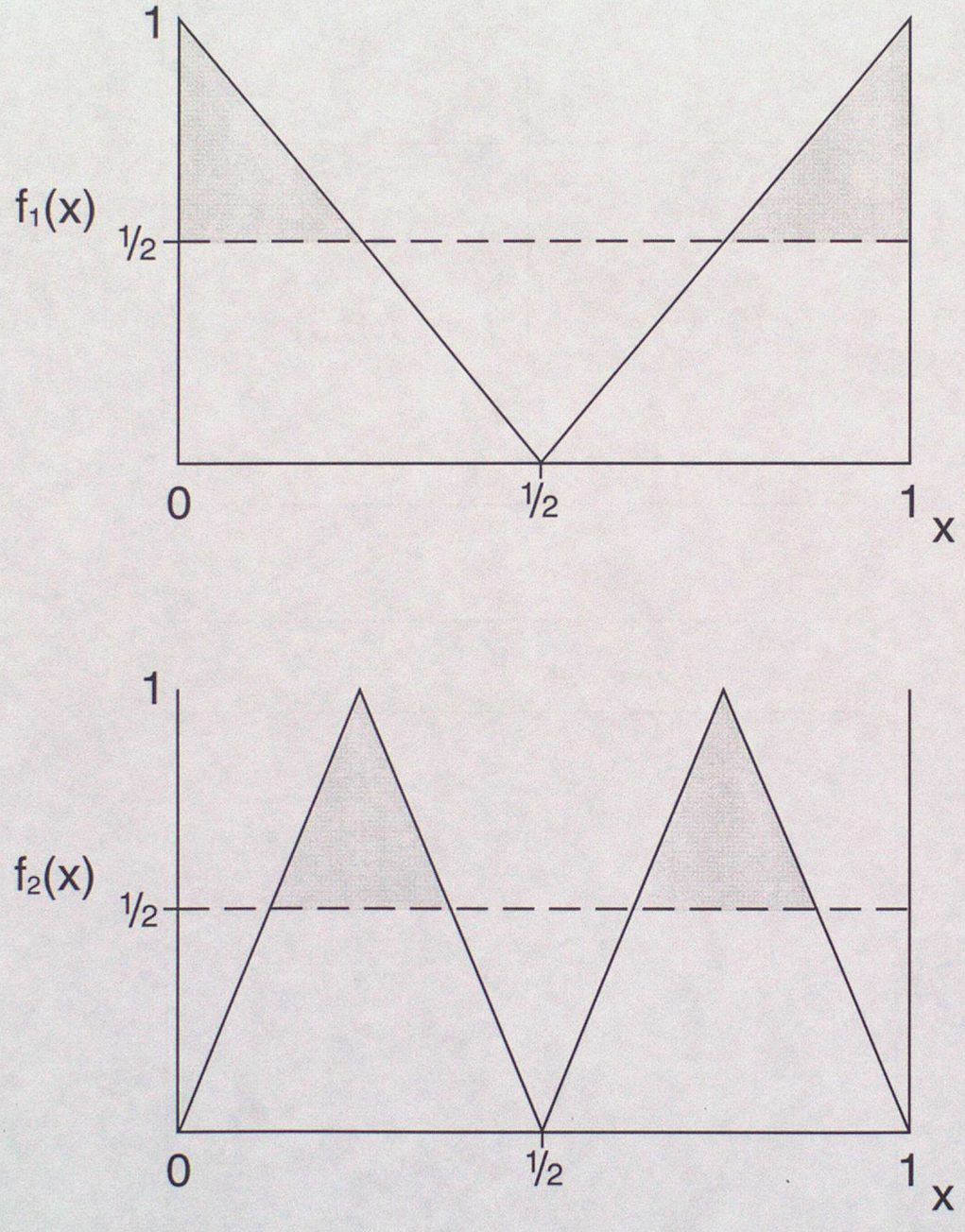
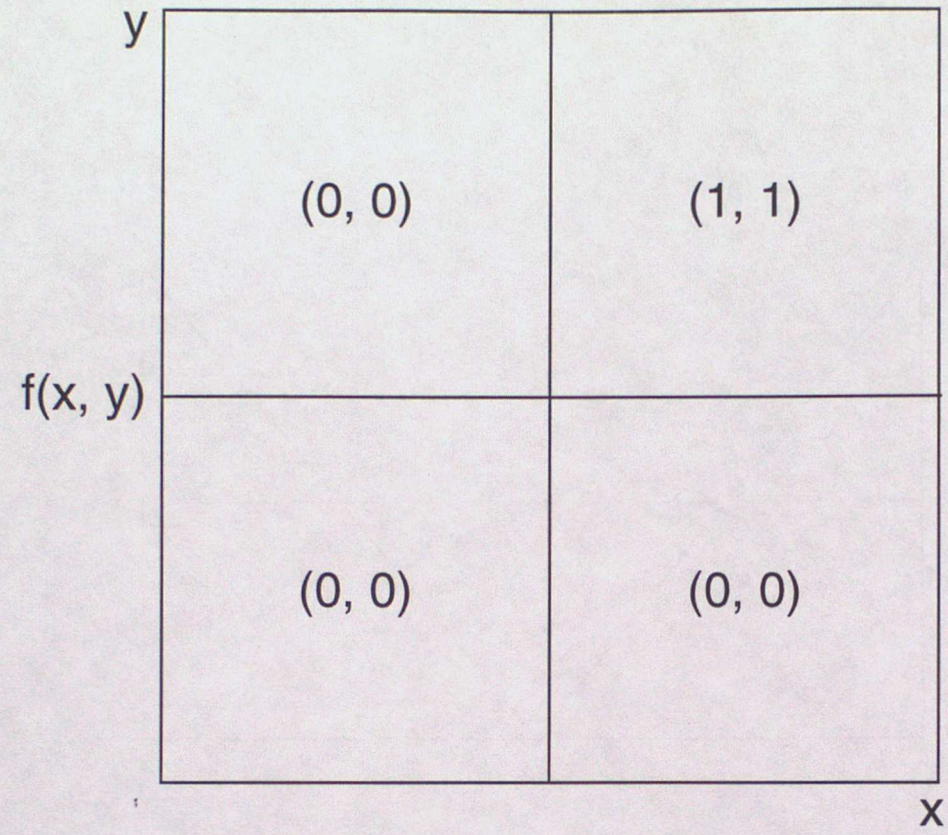


Fig.3



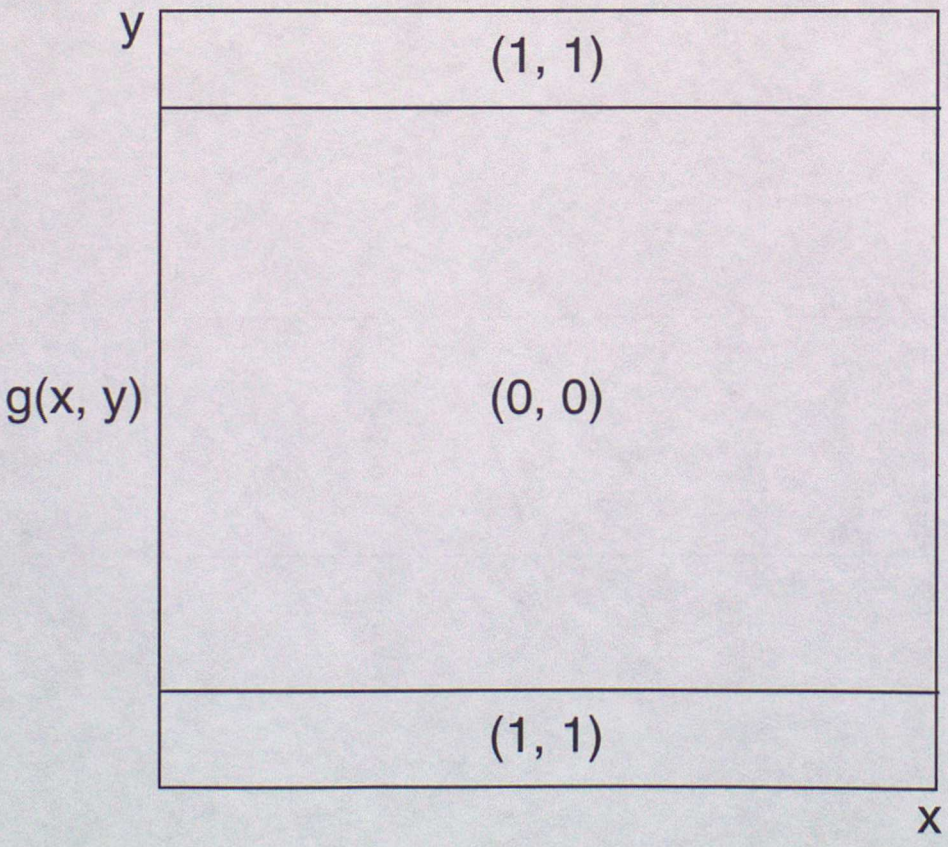


Fig.4

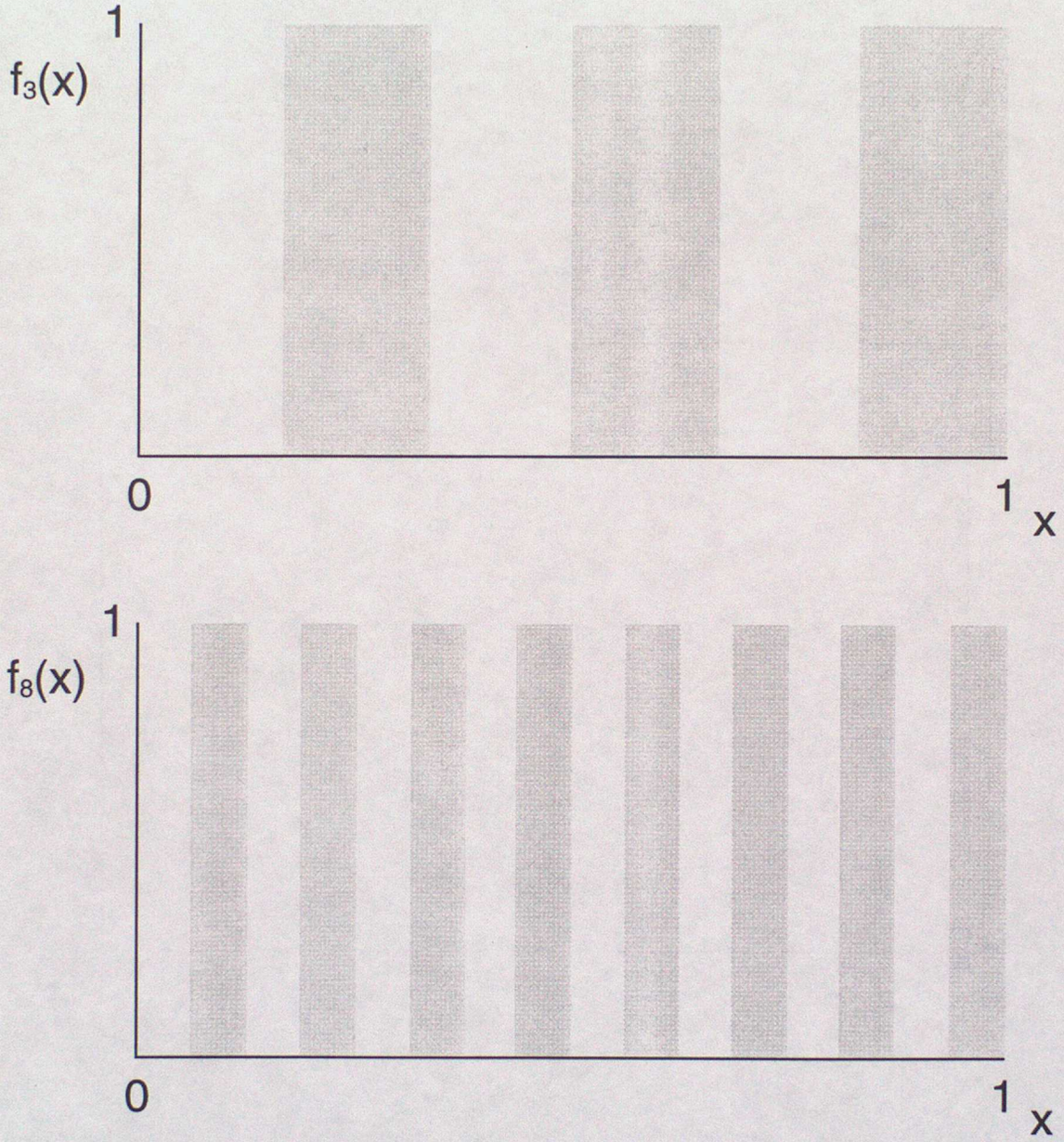


Fig.5

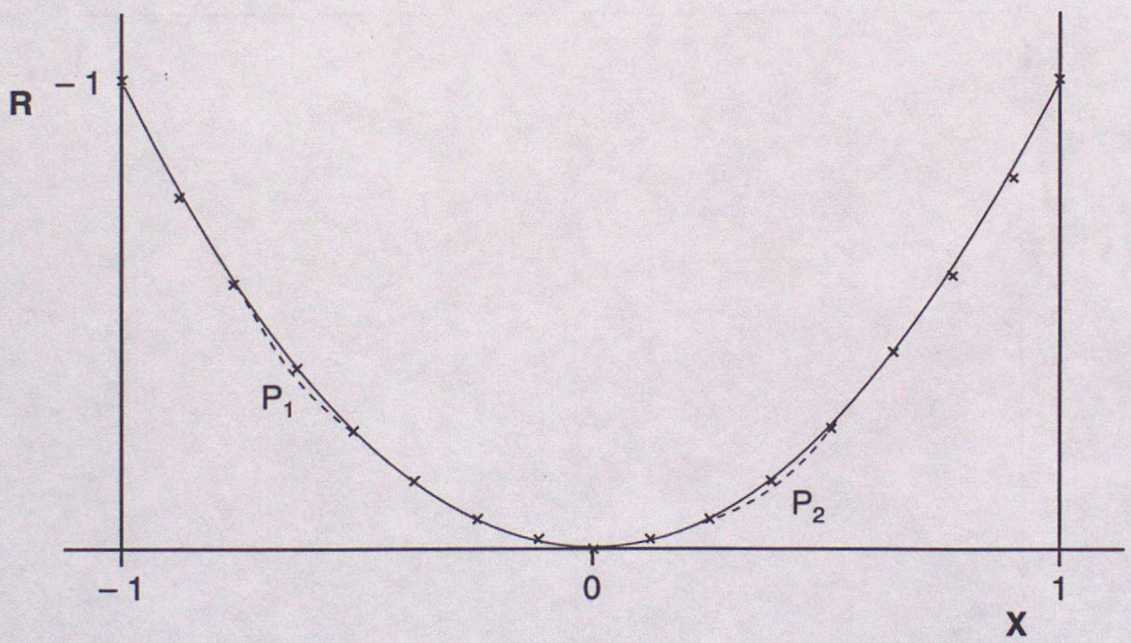


Fig.6